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Imbedding Theorems for Lipschitz Spaces Generated by the Weak-L^p Metric

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We prove, by elementary measure theoretic arguments, imbedding theorems for the Lipschitz spaces generated by the weak-L^p metric. Our results hold for every p in the range $0 < p < \infty$ and in some cases extend the results known for the L^p metric. We also show that our techniques also extend to more general situations.

The main purpose of this paper is to establish imbedding theorems for the spaces $\Lambda^{\alpha}_{L^{p,\infty}}(\mathbb{R}^n)$ of functions in weak-$L^p(\mathbb{R}^n)$ smooth up to the order α (see below for the definition). Analogous results for spaces generated by the L^p metric are well known (see, e.g., [3, Sect. 6.3]); however, the *weak*-L^p setting seems of particular interest for an elementary treatment of this subject. In fact our methods do not make use of Fourier transform or approximation techniques but only of measure theoretic properties of sets. This allows us to establish results valid for every p in the whole range $0 < p < \infty$ (see [2, Chapt. 12] for some results for the case of the L^p metric with $0 < p < 1$) and also to prove an imbedding theorem for spaces of functions defined on general measure space. Moreover since weak-$L^p(\mathbb{R}^n)$ is larger than $L^p(\mathbb{R}^n)$, in some situations our results extend the results known in the classical case (see part (iii) of Theorem 1).

We begin with some definitions.

The weak-$L^p(\mathbb{R}^n)$ space, or $L^{p,\infty}(\mathbb{R}^n)$, consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,\infty}} = [\sup_{\lambda > 0} \lambda^p \, |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \,]^{1/p}$$

is finite (we denote with $|A|$ the Lebesgue measure of a set A). A detailed exposition about these spaces can be found in [4, Chap. V].

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Let X be either the space $L^{p,\infty}(\mathbb{R}^n)$, the space $L^p(\mathbb{R}^n)$, or the space $C(\mathbb{R}^n)$ of continuous functions on $\mathbb{R}^n$ endowed with the L^∞ metric. Let $f \in X$. For any $h \in \mathbb{R}^n$ and any integer $k \geqslant 0$ the kth difference operator Δ_h^k of step h is defined by

$$\Delta_h^0 f(x) = f(x),$$
$$\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x),$$
$$\Delta_h^k f(x) = \Delta_h[\Delta_h^{k-1} f(x)] = \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} f(x+jh)$$

and the kth modulus of smoothness of f by

$$\omega_k(f, \delta)_X = \sup_{|h| \leqslant \delta} \|\Delta_h^k f\|_X.$$

Let $\alpha > 0$ and $k = [\alpha] + 1$. The *generalized* Lipschitz space $\Lambda_X^\alpha(\mathbb{R}^n)$ consists of all functions $f \in X$ for which

$$\sup_{\delta > 0} \frac{\omega_k(f, \delta)_X}{\delta^\alpha} < +\infty$$

and a *norm* in $\Lambda_X^\alpha(\mathbb{R}^n)$ is given by

$$\|f\|_{\Lambda_X^\alpha} = \|f\|_X + \sup_{\delta > 0} \frac{\omega_k(f, \delta)_X}{\delta^\alpha}.$$

When $\alpha = 0$ we set $\Lambda_X^0(\mathbb{R}^n) = X$. The Lipschitz spaces generated by functions in $C(\mathbb{R}^n)$ will be denoted simply by $\Lambda^\alpha(\mathbb{R}^n)$.

We also need the definition of the Zygmund space $L^{\exp}(\mathbb{R}^n)$.

Let f be a measurable function on $\mathbb{R}^n$. Its non-increasing rearrangement f^* is defined by

$$f^*(t) = \inf\{\lambda : |\{x : |f(x)| > \lambda\}| \leqslant t\}.$$

Roughly speaking f^* is the non-increasing function defined on $[0, \infty)$ with the same distribution as f. We also define the maximal function of f^* by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds = \sup_{|E| = t} \frac{1}{|E|} \int_E |f(x)|\, dx.$$

The space $L^{\exp}(\mathbb{R}^n)$ consists of all measurable functions f such that

$$\|f\|_{L^{\exp}} = \sup_{t > 0} \frac{f^{**}(t)}{1 + \log_+(1/t)} < +\infty. \tag{1}$$

Since

$$(af)^{**} = |a| f^{**}$$

and

$$(f+g)^{**} \leqslant f^{**} + g^{**}$$

the quantity in (1) is a norm and therefore $L^{\exp}(\mathbb{R}^n)$ a Banach space. See [1, Chap. 4, Sect. 6] for a proof of this and more properties about $L^{\exp}(\mathbb{R}^n)$.

Our main result is the following.

THEOREM 1.

(i) *Let* $0 < p < r < \infty$ *and* $\alpha \geqslant n(1/p - 1/r)$. *Then the following imbedding holds*:

$$\Lambda^{\alpha}_{L^{p,\infty}}(\mathbb{R}^n) \subset \Lambda^{\alpha - n(1/p - 1/r)}_{L^{r,\infty}}(\mathbb{R}^n).$$

(ii) *Let* $0 < p < \infty$ *and* $\alpha = n/p$. *Then the following imbedding holds*:

$$\Lambda^{\alpha}_{L^{p,\infty}}(\mathbb{R}^n) \subset L^{\exp}(\mathbb{R}^n).$$

(iii) *Let* $0 < p < \infty$ *and* $\alpha > n/p$. *Then the following imbedding holds*:

$$\Lambda^{\alpha}_{L^{p,\infty}}(\mathbb{R}^n) \subset \Lambda^{\alpha - n/p}(\mathbb{R}^n).$$

We point out that, since $\Lambda^{\alpha}_{L^p}(\mathbb{R}^n) \subset \Lambda^{\alpha}_{L^{p,\infty}}(\mathbb{R}^n)$, part (iii) of the above theorem improves the analogous result known for the L^p metric (see [3, Sect. 6.3]).

Remark. Assume $0 < p < \infty$ and $0 < \alpha < n/p$. By part (i) of the above theorem there exists a constant $C > 0$ such that

$$\|f\|_{L^{p/(1-\alpha p/n),\infty}} \leqslant C \left\{ \|f\|_{L^{p,\infty}} + \sup_{\delta > 0} \frac{\omega_k(f, \delta)_{L^{p,\infty}}}{\delta^{\alpha}} \right\}. \tag{2}$$

However, the $L^{p,\infty}$ norm of f can be dropped from the above inequality by means of a dilation argument. Let $f_\varepsilon(x) = f(x/\varepsilon)$ and note that $\|f_\varepsilon\|_{L^{p,\infty}} = \varepsilon^{n/p} \|f\|_{L^{p,\infty}}$ and that $\omega_k(f_\varepsilon, \delta)_{L^{p,\infty}} = \varepsilon^{n/p} \omega_k(f, \delta/)_{L^{p,\infty}}$. Then, applying (2) to f_ε yields

$$\|f\|_{L^{p/(1-\alpha p/n),\infty}} \leqslant C \left\{ \varepsilon^{\alpha} \|f\|_{L^{p,\infty}} + \sup_{\delta > 0} \frac{\omega_k(f, \delta)_{L^{p,\infty}}}{\delta^{\alpha}} \right\}.$$

Letting $\varepsilon \to 0$ gives, for every $f \in L^{p,\infty}(\mathbb{R}^n)$,

$$\|f\|_{L^{p/(1-\alpha p/n),\infty}} \leqslant C \sup_{\delta>0} \frac{\omega_k(f, \delta)_{L^{p,\infty}}}{\delta^\alpha}.$$

The sharpness of the embeddings of Theorem 1 follows by general consideration on the behavior of the norms under dilations and arguments similar to that used in the above remark. In the next theorem we present a more precise result which is based on the explicit computation of the moduli of smoothness of functions that behave locally like $|x|^\alpha$.

THEOREM 2. *Let* $0<p<\infty$. *For any* $\alpha>0$ *there exists a function* $f \in \Lambda^\alpha_{L^{p,\infty}}(\mathbb{R}^n)$ *such that*

(i) *for any* $r \in [p, p/(1-\alpha p/n)]$, *when* $0<\alpha<n/p$, *or any* $r \in [p, \infty)$, *when* $\alpha \geqslant n/p$, *one has* $f \in \Lambda^{\alpha - n(1/p-1/r)}_{L^{p,\infty}}(\mathbb{R}^n)$ (*by Theorem* 1) *and*

$$\omega_{[\alpha - n(1/p-1/r)]+1}(f, \delta)_{L^{r,\infty}} \geqslant c_r \delta^{\alpha - n(1/p-1/r)}$$

for a suitable constant c_r *and any* δ *sufficiently small*;

(ii) *when* $\alpha = n/p$ *one has* $f \in L^{\exp}(\mathbb{R}^n)$ (*by Theorem* 1) *and* f *is unbounded*;

(iii) *when* $\alpha > n/p$ *one has* $f \in \Lambda^{\alpha - n/p}(\mathbb{R}^n)$ (*by Theorem* 1) *and*

$$\omega_{[\alpha - n/p]+1}(f, \delta) \geqslant c\, \delta^{\alpha - n/p}.$$

The plan of the paper is the following. In the first section we state and prove a few preliminary results about the spaces $\Lambda^\alpha_{L^{p,\infty}}(\mathbb{R}^n)$. Sections 2 and 3 are devoted respectively to the proof of Theorem 1 and Theorem 2. In the last section we show that our methods also apply to a more general context. We consider Lipschitz spaces of functions defined on a measure space and we present a suitable imbedding theorem.

1. PRELIMINARY RESULTS

It is not difficult to see that the quantity

$$\|f\|_{L^{p,\infty}} = [\sup_{\lambda>0} \lambda^p \, |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|]^{1/p}$$

is not a norm for the weak-L^p spaces since the triangular inequality may fail also for $p \geqslant 1$. Even if it can be proved that for $1<p<\infty$ there exists an equivalent norm which endows a Banach space structure (see, e.g., [4, Chap. V]) in what follows we shall assume in full generality $0<p<\infty$ and we need a substitute for the triangular inequality. The next lemma provides the required substitute.

LEMMA 3. *Let* $a, b>0, a+b=1$, *and* $0<p<\infty$. *Then for every* $f, g \in L^{p,\infty}(\mathbb{R}^n)$

$$\|f+g\|^p_{L^{p,\infty}} \leqslant \frac{1}{a^p}\|f\|^p_{L^{p,\infty}} + \frac{1}{b^p}\|g\|^p_{L^{p,\infty}}. \tag{3}$$

Proof. Since

$$\{x: |f(x)+g(x)|>\lambda\} \subseteq \{x: |f(x)|>a\lambda\} \cup \{x: |g(x)|>b\lambda\}$$

a straightforward computation gives the desired result. ∎

Using (3) it is not difficult to see that if k and r are integral and $k<r$, then there exists a constant $M>0$ such that for every $f \in L^{p,\infty}(\mathbb{R}^n)$

$$\omega_r(f,\delta)_{L^{p,\infty}} \leqslant M\omega_k(f,\delta)_{L^{p,\infty}}.$$

The next proposition shows that, under suitable conditions, the above inequality can be reversed. Our proof uses an adapted version of the Marchaud inequality and follows the lines of Theorem 8.1 in [2, Chap. 2].

PROPOSITION 4. *Let* $\alpha<k<r$ *with* k *and* r *integers. Then there exists a constant* $C>0$ *such that if* $f \in L^{p,\infty}(\mathbb{R}^n)$ *satisfies*

$$\omega_r(f,\delta)_{L^{p,\infty}} \leqslant A\delta^\alpha \tag{4}$$

then

$$\omega_k(f,\delta)_{L^{p,\infty}} \leqslant CA\delta^\alpha.$$

Proof. We suppose first $r=k+1$. The general case follows iterating from this. Since

$$Q(x)=\frac{(x-1)^k-2^{-k}(x^2-1)^k}{(x-1)^{k+1}}$$

is a polynomial of degree $k-1$, replacing x with the translation operator T_h defined by $T_h f(x)=f(x+h)$ we obtain

$$(T_h-I)^k=2^{-k}(T_{2h}-I)^k+Q(T_h)(T_h-I)^{k+1}$$

that can be written as

$$\Delta_h^k f = 2^{-k} \Delta_{2h}^k f + Q(T_h)\, \Delta_h^{k+1} f$$

and therefore by (3)

$$\|\Delta_h^k f\|_{L^{p,\infty}}^p \leqslant \frac{1}{a^p} 2^{-pk} \|\Delta_{2h}^k f\|_{L^{p,\infty}}^p + \frac{1}{b^p} \|Q(T_h)\, \Delta_h^{k+1} f\|_{L^{p,\infty}}^p.$$

Using again (3) and the fact that the translation operator is bounded on $L^{p,\infty}(\mathbb{R}^n)$ we obtain

$$\|Q(T_h)\, \Delta_h^{k+1} f\|_{L^{p,\infty}} \leqslant M \|\Delta_h^{k+1} f\|_{L^{p,\infty}}$$

so that

$$\|\Delta_h^k f\|_{L^{p,\infty}}^p \leqslant \frac{1}{a^p} 2^{-pk} \|\Delta_{2h}^k f\|_{L^{p,\infty}}^p + \frac{M^p}{b^p} [\omega_{k+1}(f, |h|)_{L^{p,\infty}}]^p. \tag{5}$$

We repeatedly apply (5) to obtain, for every $m > 0$,

$$\begin{aligned} &\|\Delta_h^k f\|_{L^{p,\infty}}^p \\ &\quad \leqslant \frac{M^p}{b^p} \sum_{j=0}^{m} \frac{2^{-jpk}}{a^{jp}} [\omega_{k+1}(f, 2^j\, |h|)_{L^{p,\infty}}]^p + \frac{2^{-pk(m+1)}}{a^{p(m+1)}} \|\Delta_{2^{m+1}h}^k f\|_{L^{p,\infty}}^p \\ &\quad \leqslant \frac{M^p}{b^p} \sum_{j=0}^{m} \frac{2^{-jpk}}{a^{jp}} [\omega_{k+1}(f, 2^j|h|)_{L^{p,\infty}}]^p + \text{const}\, \frac{2^{-pk(m+1)}}{a^{p(m+1)}} \|f\|_{L^{p,\infty}}^p. \end{aligned}$$

Assuming $2^{-k} < a < 1$ and letting $m \to \infty$ we have

$$\|\Delta_h^k f\|_{L^{p,\infty}}^p \leqslant \frac{M^p}{b^p} \sum_{j=0}^{\infty} \frac{2^{-jpk}}{a^{jp}} [\omega_{k+1}(f, 2^j\, |h|)_{L^{p,\infty}}]^p$$

so that

$$[\omega_k(f, \delta)_{L^{p,\infty}}]^p \leqslant \frac{M^p}{b^p} \sum_{j=0}^{\infty} \frac{2^{-jpk}}{a^{jp}} [\omega_{k+1}(f, 2^j\delta)_{L^{p,\infty}}]^p.$$

The assumption (4) yields

$$[\omega_k(f, \delta)_{L^{p,\infty}}]^p \leqslant A^p \frac{M^p}{b^p} \sum_{j=0}^{\infty} \frac{2^{-jpk}}{a^{jp}} (2^j\delta)^{\alpha p} = A^p\, \delta^{\alpha p} \frac{M^p}{b^p} \sum_{j=0}^{\infty} (2^{pk-\alpha p} a^p)^{-j}.$$

Since $\alpha < k$ we can choose a such that $2^{\alpha - k} < a < 1$. In this way the last series converges and therefore

$$\omega_k(f, \delta)_{L^{p,\infty}} \leqslant \text{const } A\delta^\alpha. \quad \blacksquare$$

COROLLARY 5. *Let $0 < \alpha < \beta$ and $0 < p < \infty$. Then we have the imbedding*

$$\Lambda^\beta_{L^{p,\infty}}(\mathbb{R}^n) \subseteq \Lambda^\alpha_{L^{p,\infty}}(\mathbb{R}^n).$$

Proof. Let $k = [\alpha] + 1$ and $s = [\beta] + 1$. Elementary arguments show that if $f \in \Lambda^\beta_{L^{p,\infty}}(\mathbb{R}^n)$ then

$$\omega_s(f, \delta)_{L^{p,\infty}} \leqslant \text{const } \|f\|_{\Lambda^\beta_{L^{p,\infty}}} \delta^\alpha.$$

If $k = s$ this gives

$$\|f\|_{\Lambda^\alpha_{L^{p,\infty}}} \leqslant \text{const } \|f\|_{\Lambda^\beta_{L^{p,\infty}}}.$$

In the general case we can use Proposition 4 to obtain

$$\omega_k(f, \delta)_{L^{p,\infty}} \leqslant \text{const } \|f\|_{\Lambda^\beta_{L^{p,\infty}}} \delta^\alpha.$$

and therefore the desired result. $\blacksquare$

2. PROOF OF THEOREM 1

We sketch here the main idea of the proof of Theorem 1. Suppose $n = 1$, $\alpha < 1$, and $f \in \Lambda^\alpha_{L^{p,\infty}}(\mathbb{R})$ with $\|f\|_{\Lambda^\alpha_{L^{p,\infty}}} \leqslant 1$. Let $A_\lambda = \{x\colon |f(x)| > \lambda\}$. Then

$$\{x\colon |\Delta_h f(x)| > \lambda/2\} \supseteq \{x\colon |f(x+h)| > \lambda\} \cap \{x\colon |f(x)| \leqslant \lambda/2\}.$$

that is,

$$\{x\colon |\Delta_h f(x)| > \lambda/2\} \supseteq (A_\lambda - h) \cap (A_{\lambda/2})^c.$$

Suppose now that $A_{\lambda/2}$ and A_λ are intervals. Since $A_\lambda \subseteq A_{\lambda/2}$ taking $h = |A_{\lambda/2}|$ we have $(A_\lambda - h) \cap (A_{\lambda/2}) = \varnothing$ and therefore

$$(A_\lambda - h) \cap (A_{\lambda/2})^c = (A_\lambda - h).$$

It follows that

$$|A_\lambda| = |A_\lambda - h| \leqslant |\{x: |\Delta_h f(x)| > \lambda/2\}| \leqslant (\lambda/2)^{-p} \, \|\Delta_h f\|^p_{L^{p,\infty}}$$
$$\leqslant (\lambda/2)^{-p} \, |h|^{p\alpha} = (\lambda/2)^{-p} \, |A_{\lambda/2}|^{p\alpha}. \tag{6}$$

Assuming $\alpha p < 1$ and iterating the above relation between $|A_\lambda|$ and $|A_{\lambda/2}|$ we obtain

$$|A_{2^j}| \leqslant \text{const } 2^{-jp/(1-\alpha p)}$$

and therefore that $f \in L^{p/(1-\alpha p), \infty}(\mathbb{R})$.

The plan of the proof is as follows.

In Lemma 7 we establish a refined version of the recurrence relation (6). The main problem in doing this is due to the fact that in the general situation the sets A_λ are not intervals and therefore they cannot be disjoint by means of translations. However, in Lemma 6 we show that it is possible to disjoin them up to a set of small measure.

In Lemma 8 we prove the imbedding

$$\Lambda^\alpha_{L^{p,\infty}}(\mathbb{R}^n) \subset L^{p/(1-\alpha p/n), \infty}(\mathbb{R}^n) \tag{7}$$

and in Lemma 10 the imbedding

$$\Lambda^\alpha_{L^{p,\infty}}(\mathbb{R}^n) \subset \Lambda^{\alpha - n(1/p - 1/r)}_{L^{r,\infty}}(\mathbb{R}^n). \tag{8}$$

To avoid the use of high order difference operators, (7) is proved only for $\alpha < 1$ and (8) for $n(1/p - 1/r) < \min(\alpha, 1)$. The proof of part (i) of Theorem 1 then follows applying repeatedly the imbedding (8).

Part (ii) of the theorem follows iterating directly the relation obtained in Lemma 7.

Part (iii) is similar to part (i). We first prove the imbedding when $\alpha < 1$ (see Lemma 11) and then we extend the embedding to the general situation.

LEMMA 6. *Let F and G be measurable subsets of $\mathbb{R}^n$ and let Γ_n denote the volume of the unit ball in $\mathbb{R}^n$. Then for every $\varepsilon > 0$ there exists $h \in \mathbb{R}^n$ such that*

$$|h| \leqslant \left(\frac{|G|}{\varepsilon \Gamma_n} \right)^{1/n} \tag{9}$$

and

$$|(F+h) \cap G| \leqslant \varepsilon \, |F|. \tag{10}$$

Proof. Assume $|F|>0$, otherwise (10) is trivial and let $g(h)=|(F+h)\cap G|$ and $E_\varepsilon=\{h\in\mathbb{R}^n\colon g(h)>\varepsilon\,|F|\}$. Since g is continuous E_ε is measurable and

$$\varepsilon\,|F|\,|E_\varepsilon|<\int_{E_\varepsilon} g(h)\,dh\leqslant\int_{\mathbb{R}^n}|(F+h)\cap G|\,dh$$

$$=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\chi_F(x-h)\,\chi_G(x)\,dx\,dh=|F|\,|G|$$

so that

$$|E_\varepsilon|<\frac{|G|}{\varepsilon}.$$

Let $B=\{x\in\mathbb{R}^n\colon |x|\leqslant(|G|/(\varepsilon\Gamma_n))^{1/n}\}$. Since $|B|=|G|/\varepsilon$ it follows there exists $h\in B$ satisfying (10). ∎

LEMMA 7. *Let $f\in L^{p,\infty}(\mathbb{R}^n)$ and suppose there exist $\alpha>0$ and $M>0$ such that*

$$\|\Delta_h f\|_{L^{p,\infty}}\leqslant M\,|h|^\alpha$$

for every $h\in\mathbb{R}^n$. Then there exists a constant $C=C(\alpha,p,n)$ such that for every $\lambda,\mu>0$

$$|\{x\colon |f(x)|>\lambda+\mu\}|\leqslant\frac{M^p}{\lambda^p}\,|\{x\colon |f(x)|>\mu\}|^{\alpha p/n}. \tag{11}$$

Proof. Observe that

$$\{x\colon |\Delta_h f(x)|>\lambda\}\supseteq\{x\colon |f(x+h)|>\lambda+\mu\}\cap\{x\colon |f(x)|\leqslant\mu\},$$

hence setting

$$A_\lambda=\{x\in\mathbb{R}^n\colon |f(x)|>\lambda\}$$

and

$$B_\lambda(h)=\{x\in\mathbb{R}^n\colon |\Delta_h f(x)|>\lambda\}$$

we have the relation

$$B_\lambda(h)\supseteq(A_{\lambda+\mu}-h)\cap(A_\mu)^c. \tag{12}$$

We choose $h \in \mathbb{R}^n$ in the following way. For fixed $\lambda, \mu > 0$ we apply Lemma 6 with $\varepsilon = \frac{1}{2}$ to the sets $A_{\lambda+\mu}$ and A_μ to ensure there exists h, $|h| < (2\,|A_\mu|/\Gamma_n)^{1/n}$ such that

$$|(A_{\lambda+\mu} - h) \cap A_\mu| \leqslant \tfrac{1}{2}\,|A_{\lambda+\mu}|. \tag{13}$$

With this choice of h, we have

$$\begin{aligned} |B_\lambda(h)| &\geqslant |(A_{\lambda+\mu} - h) \cap (A_\mu)^c| = |A_{\lambda+\mu}| - |(A_{\lambda+\mu} - h) \cap (A_\mu)| \\ &\geqslant |A_{\lambda+\mu}| - \tfrac{1}{2}\,|A_{\lambda+\mu}| = \tfrac{1}{2}\,|A_{\lambda+\mu}|. \end{aligned}$$

So that

$$\lambda^p\,\tfrac{1}{2}\,|A_{\lambda+\mu}| \leqslant \lambda^p\,|B_\lambda(h)| \leqslant \|\Delta_h f\|^p_{L^{p,\infty}} \leqslant M^p\,|h|^{\alpha p} \leqslant M^p(2\,|A_\mu|/\Gamma_n)^{\alpha p/n} \tag{14}$$

and therefore

$$|A_{\lambda+\mu}| \leqslant \text{const}\,\frac{M^p}{\lambda^p}\,|A_\mu|^{\alpha p/n}. \quad \blacksquare$$

LEMMA 8. *Let* $0 < p < \infty$ *and* $0 < \alpha < \min(n/p, 1)$. *Then the following imbedding holds*:

$$\Lambda^\alpha_{L^{p,\infty}}(\mathbb{R}^n) \subset L^{p/(1-\alpha p/n),\,\infty}(\mathbb{R}^n).$$

Proof. Let $f \in \Lambda^\alpha_{L^{p,\infty}}(\mathbb{R}^n)$, so that

$$\|\Delta_h f\|_{L^{p,\infty}} \leqslant \|f\|_{\Lambda^\alpha_{L^{p,\infty}}}\,|h|^\alpha. \tag{15}$$

Setting

$$A_\lambda = \{x : |f(x)| > \lambda\}$$

and applying Lemma 7 we obtain

$$|A_{\lambda+\mu}| \leqslant C\,\|f\|^p_{\Lambda^\alpha_{L^{p,\infty}}}\,\frac{|A_\mu|^{\alpha p/n}}{\lambda^p}.$$

We can assume with no loss of generality

$$\|f\|^p_{\Lambda^\alpha_{L^{p,\infty}}} = 1/C \tag{16}$$

so that, taking $\lambda = \mu$,

$$|A_{2\lambda}| \leqslant \frac{|A_\lambda|^{\alpha p/n}}{\lambda^p}.$$

Iterating the above inequality yields

$$|A_{2^j}| \leqslant \frac{1}{2^{p((j-1)+(j-2)\gamma+(j-3)\gamma^2+\cdots+\gamma^{j-2})}} |A_1|^{\gamma^j},$$

where $\gamma = \alpha p/n$. Since $\gamma \neq 1$

$$(j-1)+(j-2)\gamma+(j-3)\gamma^2+\cdots+\gamma^{j-2} = \frac{\gamma^j-1}{(\gamma-1)^2}+\frac{j}{1-\gamma} \tag{17}$$

and

$$\begin{aligned} |A_{2^j}| &\leqslant 2^{-p((\gamma^j-1)/(\gamma-1)^2+j/(1-\gamma))} |A_1|^{\gamma^j} \\ &\leqslant D(C, p, \gamma) \max(1, |A_1|)\, 2^{-j(p/(1-\gamma))}. \end{aligned}$$

Moreover the assumption (16) implies $|A_1| \leqslant$ const so that

$$\|f\|_{L^{p/(1-\alpha p/n)}, \infty} \leqslant \text{const.} \quad \blacksquare$$

Lemma 9. *Let $0<p<\infty$, $0<\alpha<\beta$, and $f \in \Lambda^\beta_{L^{p,\infty}}(\mathbb{R}^n)$. For every $h \in \mathbb{R}^n$ set $g_h = |h|^{\alpha-\beta} \Delta_h^{[\beta]+1} f$. Then there exists a constant $c>0$, independent of h, such that*

$$\|g_h\|_{\Lambda^\alpha_{L^{p,\infty}}} \leqslant c\, \|f\|_{\Lambda^\beta_{L^{p,\infty}}}.$$

Proof. Observe that since $f \in \Lambda^\beta_{L^{p,\infty}}(\mathbb{R}^n)$, by Corollary 5, $f \in \Lambda^{\beta-\alpha}_{L^{p,\infty}}(\mathbb{R}^n)$ and therefore

$$\begin{aligned} \|g_h\|_{L^{p,\infty}} = |h|^{\alpha-\beta} \|\Delta_h^{[\beta]+1} f(x)\|_{L^{p,\infty}} &\leqslant \text{const}\, |h|^{\alpha-\beta} \|\Delta_h^{[\beta-\alpha]+1} f(x)\|_{L^{p,\infty}} \\ &\leqslant \text{const}\, \|f\|_{\Lambda^{\beta-\alpha}_{L^{p,\infty}}} \leqslant \text{const}\, \|f\|_{\Lambda^\beta_{L^{p,\infty}}}. \end{aligned}$$

Moreover for $\|\Delta_u^{[\beta]+1} g_h\|_{L^{p,\infty}}$ we have the estimates

$$\begin{aligned} \|\Delta_u^{[\beta]+1} g_h\|_{L^{p,\infty}} &= |h|^{\alpha-\beta} \|\Delta_u^{[\beta]+1} \Delta_h^{[\beta]+1} f\|_{L^{p,\infty}} \\ &\leqslant \text{const}\, |h|^{\alpha-\beta} \|\Delta_h^{[\beta]+1} f\|_{L^{p,\infty}} \\ &\leqslant \text{const}\, |h|^{\alpha-\beta} |h|^\beta \|f\|_{\Lambda^\beta_{L^{p,\infty}}} \end{aligned}$$

and

$$\|\Delta_u^{[\beta]+1} g_h\|_{L^{p,\infty}} = |h|^{\alpha-\beta} \|\Delta_u^{[\beta]+1} \Delta_h^{[\beta]+1} f\|_{L^{p,\infty}}$$
$$\leqslant \text{const } |h|^{\alpha-\beta} \|\Delta_h^{[\beta]+1} f\|_{L^{p,\infty}}$$
$$\leqslant \text{const } |h|^{\alpha-\beta} |u|^{\beta} \|f\|_{\Lambda^{\beta}_{L^{p,\infty}}}$$

Using the first estimate when $|u| \geqslant |h|$ and the second when $|u| < |h|$ we obtain

$$\sup_{\delta>0} \frac{\omega_{[\beta]+1}(g_h, \delta)_{L^{p,\infty}}}{\delta^{\alpha}} \leqslant \text{const } \|f\|_{\Lambda^{\beta}_{L^{p,\infty}}}.$$

If $[\beta] = [\alpha]$ the above inequality gives

$$\|g_h\|_{\Lambda^{\alpha}_{L^{p,\infty}}} \leqslant \text{const } \|f\|_{\Lambda^{\beta}_{L^{p,\infty}}}. \tag{18}$$

Otherwise we can apply Proposition 4 to obtain an estimate for $\omega_{[\alpha]+1}(g_h, \delta)$ and therefore (18). ∎

LEMMA 10. *Let* $0 < p < r < \infty$, *and* $n(1/p - 1/r) < \min(\alpha, 1)$. *Then the following imbedding holds*:

$$\Lambda^{\alpha}_{L^{p,\infty}}(\mathbb{R}^n) \subset \Lambda^{\alpha - n(1/p - 1/r)}_{L^{r,\infty}}(\mathbb{R}^n).$$

Proof. Let $f \in \Lambda^{\alpha}_{L^{p,\infty}}(\mathbb{R}^n)$, $k = [\alpha] + 1$, and $\gamma = n(1/p - 1/r)$. We fix $h \in \mathbb{R}^n$ and set $g_h(x) = |h|^{\gamma-\alpha} \Delta_h^k f(x)$. By Lemma 9 we have $g_h \in \Lambda^{\gamma}_{L^{p,\infty}}(\mathbb{R}^n)$ and $\|g_h\|_{\Lambda^{\gamma}_{L^{p,\infty}}} \leqslant \text{const } \|f\|_{\Lambda^{\alpha}_{L^{p,\infty}}}$.

Since, by Lemma 8, $\Lambda^{\gamma}_{L^{p,\infty}}(\mathbb{R}^n) \subset L^{r,\infty}(\mathbb{R}^n)$ we have

$$\|g_h\|_{L^{r,\infty}} \leqslant \text{const } \|g_h\|_{\Lambda^{\gamma}_{L^{p,\infty}}} \leqslant \text{const } \|f\|_{\Lambda^{\alpha}_{L^{p,\infty}}}.$$

It follows that

$$\|\Delta_h^k f\|_{L^{r,\infty}} \leqslant \text{const } \|f\|_{\Lambda^{\alpha}_{L^{p,\infty}}} |h|^{\alpha-\gamma}$$

and applying Proposition 4 (only in the case $[\alpha - \gamma] < [\alpha]$) we obtain

$$\sup_{\delta>0} \frac{\omega_{[\alpha-\gamma]+1}(f, \delta)_{L^{r,\infty}}}{\delta^{\alpha-\gamma}} \leqslant \text{const } \|f\|_{\Lambda^{\alpha}_{L^{p,\infty}}}.$$

To conclude the proof we only need an estimate of $\|f\|_{L^{r,\infty}}$ in terms of $\|f\|_{\Lambda^{\alpha}_{L^{p,\infty}}}$. This follows by Lemma 8 and Corollary 5. Indeed,

$$\|f\|_{L^{r,\infty}} \leqslant \text{const } \|f\|_{\Lambda^{\gamma}_{L^{p,\infty}}} \leqslant \text{const } \|f\|_{\Lambda^{\alpha}_{L^{p,\infty}}}. \quad ∎$$

Proof of Theorem 1. (i) Let $\gamma = n(1/p - 1/r)$. We have already proved the imbedding when $\gamma < 1$ in Lemma 10, so we can assume $\gamma \geqslant 1$. Let $k = [\gamma] + 1$, $\gamma_j = \gamma(j/k)$, and r_j such that $\gamma_j = n(1/p - 1/r_j)$.

Applying k times Lemma 10 we obtain the following chain of imbeddings

$$\begin{aligned} \Lambda^{\alpha}_{L^{p,\infty}}(\mathbb{R}^n) &\subset \Lambda^{\alpha-\gamma_1}_{L^{r_1,\infty}}(\mathbb{R}^n) \subset \Lambda^{\alpha-\gamma_2}_{L^{r_2,\infty}} \subset \cdots \subset \Lambda^{\alpha-\gamma_k}_{L^{r_k,\infty}}(\mathbb{R}^n) \\ &= \Lambda^{\alpha - n(1/p-1/r)}_{L^{r,\infty}}(\mathbb{R}^n). \quad \blacksquare \end{aligned}$$

Proof of Theorem 1. (ii) We assume, with no loss of generality, $\|f\|_{\Lambda^{n/p}_{L^{p,\infty}}} = 1$. Moreover observe that we can also assume $p > n$ otherwise we can apply part (i) of the theorem to ensure that $f \in \Lambda^{n/r}_{L^{r,\infty}}(\mathbb{R}^n)$ for some $r > n$. With this assumption only first order differences are involved, so that

$$\|\Delta_h f\|_{L^{p,\infty}} \leqslant |h|^{n/p}$$

and, setting

$$A_\lambda = \{x\colon |f(x)| > \lambda\},$$

by Lemma 7 we get

$$|A_{\lambda+\mu}| \leqslant \frac{C}{\lambda^p} |A_\mu|.$$

Choosing λ sufficiently large we obtain

$$|A_{\lambda+\mu}| \leqslant \frac{1}{e} |A_\mu|.$$

and therefore

$$|A_{k\lambda}| \leqslant e^{-k+1} |A_\lambda| \leqslant \text{const}\, e^{-k}.$$

From this it follows easily that

$$f^*(t) \leqslant \begin{cases} \xi \log \dfrac{c}{t} & \text{if} \quad t < \dfrac{c}{e} \\ \xi & \text{if} \quad t \geqslant \dfrac{c}{e} \end{cases}$$

for suitable positive constants c and ξ. Hence $\|f\|_{L^{\exp}} \leqslant \text{const}$. $\blacksquare$

Lemma 11. *Assume* $0<p<\infty$ *and* $n/p<\alpha<1$. *Then the following embedding holds*:

$$\Lambda^{\alpha}_{L^{p,\infty}}(\mathbb{R}^n) \subset \Lambda^{\alpha-n/p}(\mathbb{R}^n).$$

Proof. Let $f \in \Lambda^{\alpha}_{L^{p,\infty}}(\mathbb{R}^n)$ for some $\alpha > n/p$, fix $h \in \mathbb{R}^n$, and let $f_h = \Delta_h, f$. Then for every $u \in \mathbb{R}^n$

$$\|\Delta_u f_h\|_{L^{p,\infty}} \leqslant M\, |u|^{\alpha}\, \|f\|_{\Lambda^{\alpha}_{L^{p,\infty}}}$$

and

$$\|f_h\|_{L^{p,\infty}} \leqslant K\, |h|^{\alpha}\, \|f\|_{\Lambda^{\alpha}_{L^{p,\infty}}} \tag{19}$$

for suitable constants M and K. Setting $F(\lambda) = |\{x : |f_h(x)| > \lambda\}|$ by Lemma 7 we have

$$F(\lambda+\mu) \leqslant C \frac{M^p\, \|f\|^p_{\Lambda^{\alpha}_{L^{p,\infty}}}}{\lambda^p} F(\mu)^{\alpha p/n}.$$

We assume, with no loss of generality,

$$CM^p\, \|f\|^p_{\Lambda^{\alpha}_{L^{p,\infty}}} \leqslant 1 \qquad \text{and} \qquad K\, \|f\|_{\Lambda^{\alpha}_{L^{p,\infty}}} \leqslant 1 \tag{20}$$

so that

$$F(\lambda+\mu) \leqslant \frac{F(\mu)^{\gamma}}{\lambda^p}$$

with $\gamma = \alpha p/n > 1$. Then, iterating the above inequality, we obtain

$$F(u+u2^{-1}+\cdots+u2^{-k}) \leqslant \frac{F(u)^{\gamma^k}}{u^{p[1+\gamma+\cdots+\gamma^{k-1}]}2^{-p[\gamma^{k-1}+2\gamma^{k-2}+\cdots+k\gamma^0]}}$$

and, by (17),

$$\begin{aligned} &F(u+u2^{-1}+\cdots+u2^{-k}) \\ &\quad \leqslant u^{-p(\gamma^k-1)/(\gamma-1)} 2^{p[(\gamma^{k+1}-1)/(\gamma-1)^2-(k+1)/(\gamma-1)]}\, F(u)^{\gamma^k} \\ &\quad \leqslant u^{p/(\gamma-1)} \left[\frac{2^{p\gamma/(\gamma-1)^2}\, F(u)}{u^{p/(\gamma-1)}}\right]^{\gamma^k}. \end{aligned}$$

Using (19) and (20) we have

$$F(u) \leqslant u^{-p}\, \|f_h\|^p_{L^{p,\infty}} \leqslant u^{-p}\, |h|^{\alpha p}$$

so that

$$F(u+u2^{-1}+\cdots+u2^{-k})\leqslant u^{p/(\gamma-1)}[2^{p\gamma/(\gamma-1)^2}u^{-p\gamma/(\gamma-1)}\,|h|^{\alpha p}]^{\gamma^k}.$$

Taking $u=c\,|h|^{\alpha-n/p}$ with c sufficiently large we obtain

$$F(u+u2^{-1}+\cdots+u2^{-k})\leqslant u^{p/(\gamma-1)}[\tfrac{1}{2}]^{\gamma^k}.$$

Letting $k\to\infty$ and observing that F is positive and not increasing we get

$$F(2c\,|h|^{\alpha-n/p})=0,$$

that is, $\|f_h\|_{L^\infty}=\|\Delta_h f\|_{L^\infty}\leqslant 2c\,|h|^{\alpha-n/p}$. A standard approximation argument shows that f is continuous and therefore that

$$f\in\Lambda^{\alpha-n/p}(\mathbb{R}^n). \quad \blacksquare$$

Proof of Theorem 1. (iii). The case $\alpha<1$ is included in the above lemma; we consider the case $\alpha\geqslant 1$. Let now $f\in\Lambda^{\alpha}_{L^p,\infty}(\mathbb{R}^n)$ with $\alpha>n/p$ and observe that, as in the proof of (ii), we can assume $n/p<1$. To avoid the use of high order differences we reduce ourselves to the case $\alpha<1$. To do this we fix $\varepsilon>0$ such that $n/p+\varepsilon<1$ and we consider

$$g_h=|h|^{n/p+\varepsilon-\alpha}\,\Delta_h^{[\alpha]+1}\,f.$$

By Lemma 9, $g_h\in\Lambda^{n/p+\varepsilon}_{L^p,\infty}(\mathbb{R}^n)$ and by Lemma 11, $g_h\in\Lambda^{\varepsilon}(\mathbb{R}^n)$ with

$$\|g_h\|_{\Lambda^\varepsilon}\leqslant\text{const}\,\|f\|_{\Lambda^{\alpha}_{L^p,\infty}}.$$

Then

$$\|\Delta_h^{[\alpha]+2}\|_{L^\infty}=\|\Delta_h\,g_h\|_{L^\infty}\,|h|^{\alpha-n/p-\varepsilon}\leqslant\|\Delta g_h\|_{\Lambda^\varepsilon}\,|h|^{\alpha-n/p}$$
$$\leqslant\text{const}\,\|f\|_{\Lambda^{\alpha}_{L^p,\infty}}\,|h|^{\alpha-n/p}. \quad \blacksquare$$

3. PROOF OF THEOREM 2

To prove Theorem 2 we consider the family of functions

$$\varphi_\sigma(x)=\begin{cases}|x|^\sigma\log(|x|) & \text{if }\sigma\text{ is a non-negative even integer,}\\ |x|^\sigma & \text{otherwise.}\end{cases}$$

Since these functions are essentially homogeneous of degree σ the study of their regularity is particularly simple. Unfortunately φ_σ has not the right behavior at infinity. To avoid this problem we fix a vector $v \in \mathbb{R}^n$ and we consider $\Delta_v^k \varphi_\sigma$ which, for suitable values of k, shows a better behavior at infinity.

One might ask why the definition of φ_σ changes when σ is a non-negative even integer. The reason appears clear if one observes that in such a case $\Delta_v^k |x|^\sigma \equiv 0$ when $k > \sigma$.

Fix now $0 < p < \infty$, $\alpha \geqslant 0$, $k = [\alpha] + 1$, and let $f = \Delta_v^k \varphi_{\alpha - n/p}$. In the next two lemmas we compute explicitly the moduli of smoothness of f. This will prove Theorem 2.

LEMMA 12. *Let f as above and $p \leqslant r < p/(1 - \alpha p/n)$ if $\alpha < n/p$ or $r \geqslant p$ if $\alpha \geqslant n/p$. Then $f \in \Lambda_{L^{r,\infty}}^{\alpha - n(1/p - 1/r)}(\mathbb{R}^n)$. Moreover*

$$\omega_{[\alpha - n(1/p - 1/r)] + 1}(f, \delta)_{L^{r,\infty}} \geqslant c\, \delta^{\alpha - n(1/p - 1/r)}$$

for δ sufficiently small.

Proof. It is not difficult too see that $|f(x)| \leqslant \text{const}\, |x|^{-n/p}$ for large values of $|x|$. This implies $f \in L^{r,\infty}(\mathbb{R}^n)$. To estimate the moduli of smoothness of f let $l = [\alpha - n(1/p - 1/r)] + 1$, $\beta = \alpha - n/p$, and observe that

$$\Delta_h^l \varphi_\beta(x) = |h|^\beta \Delta_{h/|h|}^l \varphi_\beta\left(\frac{x}{|h|}\right). \tag{21}$$

From this we obtain

$$\begin{aligned}
&\|\Delta_h^l f\|_{r,\infty}^r \\
&\quad = \|\Delta_v^k \Delta_h^l \varphi_\beta\|_{r,\infty}^r \\
&\quad = \sup_{\lambda > 0} \lambda^r \left|\left\{x : \left|\sum_{j=0}^k (-1)^{j+k} \binom{k}{j} \Delta_h^l \varphi_\beta(x + jv)\right| > \lambda\right\}\right| \\
&\quad = \sup_{\lambda > 0} \lambda^r \left|\left\{x : |h|^\beta \left|\sum_{j=0}^k (-1)^{j+k} \binom{k}{j} \Delta_{h/|h|}^l \varphi_\beta\left(\frac{x + jv}{|h|}\right)\right| > \lambda\right\}\right| \\
&\quad = \sup_{\lambda > 0} \lambda^r \left|\left\{x : \left|\sum_{j=0}^k (-1)^{j+k} \binom{k}{j} \Delta_{h/|h|}^l \varphi_\beta\left(\frac{x}{|h|} + \frac{jv}{|h|}\right)\right| > \lambda\, |h|^{-\beta}\right\}\right| \\
&\quad = \sup_{\lambda > 0} \lambda^r\, |h|^n \left|\left\{x : \left|\sum_{j=0}^k (-1)^{j+k} \binom{k}{j} \Delta_{h/|h|}^l \varphi_\beta\left(x + \frac{jv}{|h|}\right)\right| > \lambda\, |h|^{-\beta}\right\}\right| \\
&\quad = |h|^{n + \beta r} \sup_{\lambda > 0} \lambda^r \left|\left\{x : \left|\sum_{j=0}^k (-1)^{j+k} \binom{k}{j} \Delta_{h/|h|}^l \varphi_\beta\left(x + \frac{jv}{|h|}\right)\right| > \lambda\right\}\right|.
\end{aligned}$$

Hence

$$\|\Delta_h^l f\|_{L^{r,\infty}} \leqslant \text{const } |h|^{\beta + n/r} \, \|\Delta_{h/|h|}^k \varphi_\beta\|_{L^{r,\infty}} \leqslant \text{const } |h|^{\alpha - n(1/p - 1/r)}.$$

To estimate $\|\Delta_h^l f\|_{L^{r,\infty}}$ from below fix a neighborhood of the origin U and observe that in U the functions $\Delta_{h/|h|}^l \varphi_\beta(x + jv/|h|)$ with $j = 1, ..., k$ are arbitrary small for small values of $|h|$. It follows that

$$\begin{aligned}
&\|\Delta_h^l f\|_{r,\infty}^r \\
&\quad = |h|^{n+\beta r} \sup_{\lambda > 0} \lambda^r \left| \left\{ x : \left| \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} \Delta_{h/|h|}^l \varphi_\beta\left(x + \frac{jv}{|h|}\right) \right| > \lambda \right\} \right| \\
&\quad \geqslant |h|^{n+\beta r} \sup_{\lambda > 0} \lambda^r \left| \left\{ x \in U : \left| \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} \Delta_{h/|h|}^l \varphi_\beta\left(x + \frac{jv}{|h|}\right) \right| > \lambda \right\} \right| \\
&\quad \geqslant |h|^{n+\beta r} \, | \{ x \in U : |\Delta_{h/|h|}^l \varphi_\beta(x)| > 1/2 \} | \geqslant \text{const } |h|^{n+\beta r}.
\end{aligned}$$

So that

$$\|\Delta_h^l f\|_{r,\infty}^r \geqslant \text{const } |h|^{\alpha - n(1/p - 1/r)}. \quad \blacksquare$$

Lemma 13. *Let* $0 < p < \infty$, $\alpha > n/p$, *and assume* f *as in the previous lemma. Then* $f \in \Lambda^{\alpha - n/p}(\mathbb{R}^n)$. *Moreover*

$$\omega_{[\alpha - n/p] + 1}(f, \delta) \geqslant c\delta^{\alpha - n/p}.$$

We omit the proof since it is similar to that of the previous lemma.

Proof of Theorem 2. Part (i) is proved in Lemma 12. As for part (ii) observe that in this case $f = \Delta_v^k \log |x|$. By Lemma 12, $f \in \Lambda_{p,\infty}^\alpha(\mathbb{R}^n)$ and by Theorem 1, $f \in L^{\exp}$. On the other hand f is clearly unbounded. Part (iii) is proved in Lemma 13.

4. AN IMBEDDING THEOREM FOR GENERALIZED LIPSCHITZ SPACES

Let $(\mathbb{X}, \mu)$ be a measure space equipped with a family of measure preserving transformation $\mathscr{T} = \{T_{\sigma,t}\}_{\sigma \in \Sigma, t \in \mathbb{R}_+}$. Every transformation $T_{\sigma,t} \in \mathscr{T}$ can be seen as a translation over $\mathbb{X}$ of a quantity t in direction σ. Following this point of view we define the first order difference operator

$$\Delta_{\sigma,t} f(x) = f(T_{\sigma,t} x) - f(x)$$

and, for $0<\alpha<1$ and $0<p<\infty$, the Lipschitz space $\Lambda^{\alpha}_{L^{p,\infty}}(\mathbb{X}, \mathscr{T})$ as the space of all measurable functions f on $\mathbb{X}$ such that

$$\|f\|_{\Lambda^{\alpha}_{L^{p,\infty}}} = \|f\|_{L^{p,\infty}} + \sup_{\sigma,t} \frac{\|\Delta_{\sigma,t} f\|_{L^{p,\infty}}}{t^{\alpha}}$$

is finite.

In the proof of Theorem 1 the fact that arbitrary sets can be disjoint up to a set of arbitrary small measure by means of translation plays a relevant role. This property is proved in Lemma 6. In the present situation we assume it. More precisely we assume there exist two constants $c, d>0$ with the following property: for every couple of measurable subsets F, G of $\mathbb{X}$ and for every $\varepsilon>0$ there exist $\sigma\in\Sigma$ and $t<c(u(G)/\varepsilon)^{1/d}$ such that

$$\mu(G\cap T_{\sigma,t}F)\leqslant \varepsilon\mu(F).$$

Observe that now d plays the role of the dimension. Under the above assumption we can prove the following.

THEOREM 14. *Let* $0<p<\infty$, *and* $0<\alpha<\min(1, d/p)$. *Then the following imbedding holds*:

$$\Lambda^{\alpha}_{L^{p,\infty}}(\mathbb{X}, \mathscr{T})\subset L^{p/(1-\alpha p/d),\infty}(\mathbb{X}).$$

Proof. First observe that under the assumptions on $\mathscr{T}$ it is possible to prove a version of Lemma 7 adapted to the present situation. The proof of the theorem now follows the same line of the proof of Lemma 8. We omit the details.

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